PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Combinatorial Analysis. The Foundations of a New Theory

P. A. MacMahon

Phil. Trans. R. Soc. Lond. A 1900 **194**, 361-386
doi: 10.1098/rsta.1900.0021

IX. *Combinatorial Analysis. The Foundations of a New Theory.*

By Major P. A. MacMahon, *D.Sc., F.R.S.*

Received March 19,—Read April 5, 1900.

Introduction.

In the 'Transactions of the Cambridge Philosophical Society' (vol. 16, Part IV., p. 262), I brought forward a new instrument of research in Combinatorial Analysis, and applied it to the complete solution of the great problem of the "Latin Square," which had proved a stumbling block to mathematicians since the time of Euler. The method was equally successful in dealing with a general problem of which the Latin Square was but a particular case, and also with many other questions of a similar character. I propose now to submit the method to a close examination, to attempt to establish it firmly, and to ascertain the nature of the questions to which it may be successfully applied. We shall find that it is not merely an enumerating instrument but a powerful reciprocating instrument, from which a host of theorems of algebraical reciprocity can be obtained with facility.

We will suppose that combinations defined by certain laws of combination have to be enumerated; the method consists in designing, on the one hand, an operation and, on the other hand, a function in such manner that when the operation is performed upon the function a number results which enumerates the combinations. If this can be carried out we, in general, obtain far more than a single enumeration; we arrive at the point of actually representing graphically all the combinations under enumeration, and solve by the way many other problems which may be regarded as leading up to the problem under consideration. In the case of the Latin Square it was necessary to design the operation and the function the combination of which was competent to yield the solution of the problem. It is a much easier process, and from my present standpoint more scientific, to start by designing the operation and the function, and then to ascertain the questions which the combination is able to deal with.

§ 1.

Art. 1.—I will commence by taking the simplest possible question to which the method is applicable. Let us inquire into the number of permutations of n different letters. A knowledge of the result would at once lead us to design

An operation.	A function.
$(d/dx)^n$	x^n

since $(d/dx)^n\, x^n = n\,!$; but once we observe the way in which d/dx operates upon x^n we require no previous knowledge of the result to aid us in the design. Conceive x^n written as a product

$$\overset{xx}{xxxxnn}\ .\ .\ .$$

the operation of d/dx consists in substituting unity for x in all possible ways, and summing the results obtained.

$$\frac{d}{dx}x^n = 1\,.\,xxx\ .\ .\ . + x1xx\ .\ .\ . + xx1xx\ .\ .\ . + .\ .\ . = nx^{n-1}.$$

We have, in fact, to perform n operations of substitution; let us select one of these, say—

$$xx1xx\ .\ .\ .$$

and denote the minor operation, by which it has been obtained, by the scheme

		1_a			...

the suffix a denoting that the first operation of d/dx has resulted in the appearance of the unit.

To obtain $\frac{d}{dx}(xx1xx\ .\ .\ .)$ we have $n-1$ minor operations by which x is replaced by unity in all possible ways. If one term obtained be

$$\overset{x}{1x1xn}\ .\ .\ .$$

the operations by which this has been reached may be denoted by the scheme

		1_a			...
1_b					...

and by proceeding in this manner we finally reach a lattice, square and of n^2 compartments, which is the diagrammatic representation of one of the $n\,!$ combinations of minor operations which results from the operation of $(d/dx)^n$ upon x^n. If we transfer the 1_b, 1_c, . . . to the top row we see that to each diagram corresponds a permutation of the n different letters a, b, c, . . . Moreover suppressing the letters a, b, c, . . . we see that we have solved the following problem, viz.:—To place n units in the compartments of the square of order n, so that each row and each column contains one and only one unit. In general we find that the problems that can be solved have some simple definition upon a lattice, as in the present instance. Writing a, b, c, . . . as a_1, a_2, a_3, . . . the suffix of the letter is given by the row and the place in the permutation by the column, so that to a_3 standing t^{th} in a permutation would correspond a unit in the s^{th} row and t^{th} column of the lattice.

It may be remarked, and will afterwards appear, that in general many different designs of operation and function are appropriate to a particular problem.

Art. 2.—With the immediate object of applying the method to the general case of permutation, there being any number of identities of letters, we must first obtain another solution of the foregoing problem.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots$ be a number of quantities and $a_1, a_2, a_3, \ldots$ their elementary symmetric functions. Further, let

$$d_1 = \mathrm{D}_1 = \frac{d}{da_1} + a_1\frac{d}{da_2} + a_2\frac{d}{da_3} \ldots$$

and $a_s = \Sigma\alpha_1\alpha_2 \ldots \alpha_3 = (1^s)$ in partition notation.

We may take as operation and function

$$\mathrm{D}_1^{\,n} \quad \text{and} \quad (1)^n,$$

equivalent to $(d/da_1)^n$ and $a_1^{\,n}$, which we had before, but more convenient as being readily generalisable.

Let $\mathrm{D}_s = \frac{1}{s\,!}d_1^{\,s}$, $d_1^{\,s}$ denoting an operator of order s, obtained by symbolical multiplication as in Taylor's theorem. Suppose the question be the enumeration of the permutations of the quantities in $\alpha_1^{\pi_1}\alpha_2^{\pi_2} \ldots \alpha_n^{\pi_n}$, where $\Sigma\pi = n$. I say that the operation and function are respectively

$$\mathrm{D}_{\pi_1}\mathrm{D}_{\pi_2} \ldots \mathrm{D}_{\pi_n} \quad \text{and} \quad (1)^n.$$

Observe that this is merely the multinomial theorem for

$$(1)^n = \ldots + \frac{n\,!}{\pi_1\,!\,\pi_2\,!\ldots\pi_n\,!}\Sigma\alpha_1^{\pi_1}\alpha_2^{\pi_2} \ldots \alpha_n^{\pi_n} + \ldots$$

$$= \ldots + \frac{n\,!}{\pi_1\,!\,\pi_2\,!\ldots\pi_n\,!}(\pi_1\pi_2 \ldots \pi_n) + \ldots$$

in partition notation; and

$$\mathrm{D}_{\pi_1}\mathrm{D}_{\pi_2}\mathrm{D}_{\pi_3} \ldots \mathrm{D}_{\pi_n}(\pi_1\pi_2 \ldots \pi_n) = 1.^*$$

Hence

$$\mathrm{D}_{\pi_1}\mathrm{D}_{\pi_2}\mathrm{D}_{\pi_3} \ldots \mathrm{D}_{\pi_n}(1)^n = \frac{n\,!}{\pi_1\,!\,\pi_2\,!\,\pi_3\,!\ldots\pi_n\,!}$$

the result we require.

The important operator D_π has been discussed by the author.† Its effect upon a monomial symmetric function is to erase a part π from the partition expression of the function.

Thus

$$\mathrm{D}_\pi\Sigma\alpha^\pi\beta^\rho\gamma^\sigma \ldots = \mathrm{D}_\pi(\pi\rho\sigma \ldots) = (\rho\sigma \ldots) = \Sigma\alpha^\rho\beta^\sigma \ldots$$

* See HAMMOND, 'Proc. Lond. Math. Soc.,' vol. 13, p. 79; also 'Trans. Camb. Phil. Soc.,' *loc. cit.*

† 'Messenger of Mathematics,' vol. 14, p. 164. 'American Journal of Mathematics,' "Third Memoir on a New Theory of Symmetric Functions," vol. 13, p. 8 *et seq.*, p. 34 *et seq.* 'Trans. Camb. Phil. Soc.,' vol. 16, part IV., p. 262.

If no part π presents itself in the operand, D_π causes the monomial function to vanish. Thus

$$D_\pi(\rho\sigma\tau \ . \ . \ .) = 0 .$$

The compound operation $D_{\pi_1}D_{\pi_2}D_{\pi_3} \ . \ . \ .$ denotes the *successive* performance of the operations

$$D_{\pi_1}, D_{\pi_2}, D_{\pi_3}, \ . \ . \ . \text{ of orders } \pi_1, \pi_2, \pi_3, \ . \ . \ . \text{ respectively.}$$

The law of operation of D_π establishes that the component operations $D_{\pi_1}, D_{\pi_2}, D_{\pi_3}, \ . \ . \ .$ may be performed *in any order*. Thus

$$D_{\pi_1}D_{\pi_2}(\pi_1\pi_2\rho\sigma \ . \ . \ .) = D_{\pi_1}(\pi_1\rho\sigma \ . \ . \ .) = D_{\pi_2}(\pi_2\rho\sigma \ . \ . \ .) = (\rho\sigma \ . \ . \ .).$$

As the order of operation is immaterial, it is found convenient in most cases to operate with $D_{\pi_1}D_{\pi_2}D_{\pi_3} \ . \ . \ .$ in the order $D_{\pi_1}, D_{\pi_1}, D_{\pi_3}, \ . \ . \ .$; this may seem at first sight at variance with the ordinary usage in the Differential Calculus, but there is a convenience in ordering the operator from left to right in agreement with the practice of ordering a partition from left to right. If, further, we note the result—

$$D_\pi(\pi) = 1$$

we have a complete account of the operator so far as it is concerned with an operand, which is merely a monomial symmetric function. The operation of D_π upon a symmetric function product is of even greater importance in the present theory. It has the effect of erasing a partition of π from the product, one part from each factor, in all possible ways; the result of the operation being a sum of products, one product arising from each such erasure of a partition. This has been set forth at length in the papers to which reference has been given, but in deference to the suggestion of one of the Referees appointed by the Royal Society to report upon the present paper, a number of examples are given to familiarise readers with the processes which are so much employed in what follows.

Example 1.—Consider

$$D_\pi(1)^n .$$

The operand consists of n factors, each of which is (1); the operator D_π is performed through the partition of the number π which involves π units; this partition must be erased from $(1)^n = (1)(1)(1) \ . \ . \ .$ to n factors in each of the $\binom{n}{\pi}$ possible ways, and the results added. Thus

$$D_\pi(1)^n = \binom{n}{\pi}(1)^{n-\pi} .$$

As a particular case

$$D_2(1)^4 = (\not{1})(\not{1})(1)(1) + (\not{1})(1)(\not{1})(1) + (\not{1})(1)(1)(\not{1}) + (1)(\not{1})(\not{1})(1)$$
$$+ (1)(\not{1})(1)(\not{1}) + (1)(1)(\not{1})(\not{1}) = 6(1)^2 = \binom{4}{2}(1)^2 .$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This, the simplest example that could be taken, shows clearly the great value of the operator D_π as an instrument in combinatorial analysis.

Example 2.—It follow from the first example that if $\Sigma\pi = n$,

$$D_{\pi_1}D_{\pi_2}D_{\pi_3} \ldots D_{\pi_n}(1)^n = \frac{n!}{\pi_1!\,\pi_2!\,\pi_3!\ldots\pi_n!}.$$

Example 3.—Consider

$$D_4(2)^2(1)^2.$$

We are concerned with the partitions of the number 4 into 2,2 and 2,1,1, and

$$D_4(2)(2)(1)(1) = (\not 2)(\not 2)(1)(1) + (\not 2)(2)(\not 1)(\not 1) + (2)(\not 2)(\not 1)(\not 1) = (1)^2 + 2(2).$$

If now we operate with D_2 we have to take account of the partitions 1,1 and 2 of the number 2, and we find

$$D_4D_2(2)^2(1)^2 = (\not 1)(\not 1) + 2(\not 2) = 3,$$

and we have the result

$$(2)^2(1)^2 = \ldots + 3(42) + \ldots$$

as a consequence.

Similarly, reversing the order of the operations

$$D_2(2)(2)(1)(1) = 2(2)(1)(1) + (2)(2) \quad \text{and} \quad D_4(2)(1)(1) = D_4(2)(2) = 1,$$

verifying the previous result.

If no partition of π can be picked out in this way from the partitions of the functions forming the product, the result of the operation is zero.

Example 4.—

$$D_2(1^2)^2 = D_2(1^2)(1^2) = (1)(1) = (1)^2.$$

It is important to notice here that a unit is erased from $(1^2) = (11)$ in *only one way*, and that for present purposes a number of similar figures enclosed in a bracket are to be considered as the same, and not different; we have already seen that when the figures are similar, but in different brackets, they are, for the purpose of selection, to be considered as different figures.

Observe that since

$$D_2(1^2)^2 = (1)^2 = (2) + 2(1^2),$$

we may say that

$$(1^2)^2 = (2^2) + 2(21^2) + \ldots,$$

the terms to be added on the right for the full expression being such as do not contain a figure 2.

To obtain the lattice representations, suppose $\pi_1, \pi_2, \pi_3, \ldots \pi_n$ to be in descending order, and thus to be an ordered partition of the number n.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Since $D_{\pi_1}(1)^n = \binom{n}{\pi_1}(1)^{n-\pi_1}$, D_{π_1} may be regarded as breaking up into $\binom{n}{\pi_1}$ minor operations, each of which consists in erasing π_1 of the n factors

$$(1)(1)(1) \ldots (1),$$

and replacing them by units. On the diagram we denote one such minor operation by (say),

1	1		1		1	1			---- n° of units = π_1

Keeping to this term,

$$1 . 1 . (1) . 1 . (1) . 1 . 1 . (. 1) (1) \ldots$$

and operating with D_{π_2}, we find the operation breaking up into $\binom{n-\pi_1}{\pi_2}$ minor operations, each of which consists in erasing π_2 of the factors, and replacing them by units. Selecting one of these minor operations, we find that the corresponding term has been obtained by operations conforming to the diagram of two rows

1	1		1		1	1			---- n° of units = π_1
		1					1	1	---- n° of units = π_2

Proceeding in this manner, we finally arrive at a lattice of n rows and n columns, such that there is one and only one unit in each column, while the numbers of units in the 1st, 2nd . . . nth rows are $\pi_1, \pi_2, \ldots \pi_n$ respectively.

We have thus the associated problem on the lattice, and we obtain $\dfrac{n!}{\pi_1!\,\pi_2!\ldots\pi_\pi!}$ representations on the lattice corresponding to the permutations of the quantities in $\alpha_1^{\pi_1}\alpha_2^{\pi_2} \ldots \alpha_n^{\pi_n}$.

The simple introductory examples lead one to expect that the method will be found capable of dealing with questions either of a chess board character or which are concerned with rectangular lattices. Further, the idea of lattice rotation gives promise of leading to theorems of algebraic reciprocity and of reciprocity in the theory of numbers, features almost inseparable from any lattice theory.

Art. 3.—*The graph of a partition.*—It is convenient to have before us the connexion between this theory and the Sylvester-Ferrers graph of a unipartite partition.

Consider the operation $D_3D_2^2D_1$ and the symmetric function $(1^4)(1^3)(1)$, (32^21) being the partition conjugate to (431).

$D_3D_2^2D_1\ (1^4)(1^3)(1) = D_2^2D_1\ (1^3)(1^2)$, the operation D_3 erasing one part, viz.: unity from each factor; this we denote as usual by

1	1	1

;

again $D_2{}^2D_1\,(1^3)(1^2) = D_2D_1\,(1^2)(1)$ and the two operations together give us

1	1	1
1	1	

and $D_2D_1\,(1^2)(1) = D_1\,(1) = 1$ and the complete lattice representation is

1	1	1
1	1	
1	1	
1		

which is none other than the graph of the partition (32^21) or of (431) according as it is read by rows or by columns. We might also have operated with $D_4D_3D_1$ upon $(1^3)\,(1^2)^2\,(1)$ and, in general, if $(\pi_1\pi_2\pi_3\,\ldots)$, $(\rho_1\rho_2\rho_3\,\ldots)$ be conjugate partitions, we obtain their graphs either by operating with $D_{\pi_1}D_{\pi_2}D_{\pi_3}\ldots$ upon $(1^{\rho_1})\,(1^{\rho_2})\,(1^{\rho_3})\ldots$ *or* with $D_{\rho_1}D_{\rho_2}D_{\rho_3}\ldots$ upon $(1^{\pi_1})\,(1^{\pi_2})\,(1^{\pi_3})\ldots$

Art. 4.—I proceed to consider some less obvious but equally interesting examples of the method. The diagrams obtained depend upon the law by which the operation is performed upon the function which is the operand. The operator D_π in connexion with symmetric function operands is of commanding importance. It would be difficult to imagine an operation better adapted to research in combinatorial analysis. We shall find later that an analogous operation exists which can be employed when symmetric functions of several systems of quantities are taken as operands. As an example of diagram formation, take as operator $D_4D_3{}^2$ and as function $(3)\,(21)\,(2)\,(1)\,(1)$ the weight of operator and of function being the same.

We have

$$
\begin{aligned}
&D_4(3)(21)(2)(1)(1)\\
&\quad= (\,.\,)(2.)(2)(1)(1) + (\,.\,)(21)(2)(\,.\,)(1) + (\,.\,)(21)(2)(1)(\,.\,)\\
&\qquad\qquad + (3)(.1)(\,.\,)(1)(1)\\
&\quad+ (3)(.1)(2)(\,.\,)(\,.\,) + (3)(2.)(\,.\,)(1)(\,.\,) + (3)(2.)(\,.\,)(\,.\,)(1)\\
&\qquad\qquad + (3)(21)(\,.\,)(\,.\,)(\,.\,),
\end{aligned}
$$

the eight terms arising from the partitions 31, 22, 211 of the number 4. The dots take the place of the picked out partitions.

Hence

$$
\begin{aligned}
&D_4(3)(21)(2)(1)(1)\\
&= (2)^2(1)^2 + 2(21)(2)(1) + (3)(1)^3 + 3(3)(2)(1) + (3)(21).
\end{aligned}
$$

The operation D_4 breaks up here into eight minor operations; taking any one of these—say that one which consists in taking 3 from the factor (3), and 1 from the factor (21), we form the first row of our diagram, viz.:—

3	1			

The term resulting from the selected minor operation is

$$(.)(2)(2)(1)(1);$$

the operation of D_3 results in four minor operations corresponding to the four ways of picking out a 2 and a 1 from different factors; we may select the particular minor operation which results in

$$(.)(.)(2)(1)(.),$$

and now we add on the second row which denotes this minor operation, and obtain the diagram of two rows.

3	1			
	2			1

We can now only operate in one way with D_3 upon (.) (.) (2) (1) (.) and we finally obtain the diagram of three rows:—

3	1			
	2			1
		2	1	

which possesses the property that the sums of the numbers in the successive rows are 4, 3, 3, respectively, while the successive columns involve the partitions (3), (21), (2), (1), (1) respectively.

The number of such diagrams is A where

$$(3)(21)(2)(1)^2 = \ldots + A(43^2) + \ldots;$$

and A has the analytical expression

$$D_4 D_3^2 (3)(21)(2)(1)^2.$$

Let us now consider the problem of placing units in the compartments of a lattice of m rows and l columns, not more than one unit in each compartment, in such wise that we can count $\mu_1, \mu_2, \ldots \mu_m$ units in the successive rows, and $\lambda_1, \lambda_2, \ldots \lambda_l$ units in the successive columns. Take

As operation.		As function.
$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m}$	and	$(1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l})$.

If

$$(1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l}) = \ldots + A(\mu_1 \mu_2 \ldots \mu_m) + \ldots$$

$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m} (1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l}) = A,$$

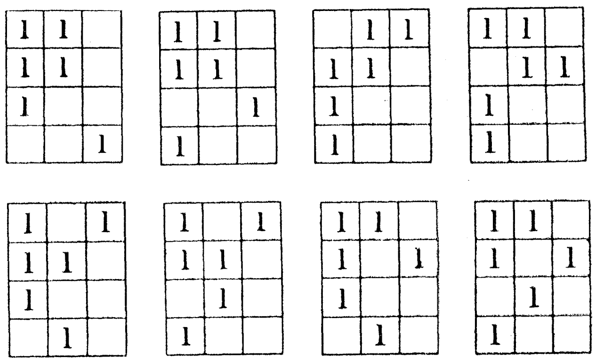
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and we can show that the number A enumerates the lattices under investigation. The operation D_{μ_1} makes selections of every μ_1 of the l factors and erases a part unity from each; one minor operation of D_{μ_1} therefore is denoted by μ_1 units placed in μ_1 compartments of the first row of a lattice of l rows; the operation D_{μ_2} adds on a second row, in which units appear in μ_2 of the compartments, and so on we finally arrive at a lattice possessing the desired property as regards rows, and as obviously the column property obtains, the problem is solved.

Ex. gr. Take $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\mu_1 = 2$, $\mu_2 = 2$, $\mu_3 = 1$, $\mu_4 = 1$

$$(1^3)(1^2)(1) = a_3 a_2 a_1 = \ldots + 8(2211) + \ldots$$

The eight diagrams are



and no others possess the desired property.

We can now apply the method so as to be an instrument of reciprocation in algebra. If we transpose the diagrams so as to read by rows as they formerly did by columns, the effect is to interchange the set of numbers $\lambda_1, \lambda_2, \ldots \lambda_l$ with the set $\mu_1, \mu_2, \ldots \mu_m$, and the number of diagrams is not altered. Hence the reciprocal theorem.

If $$(1^{\lambda_1})(1^{\lambda_2}) \ldots (1^{\lambda_l}) = \ldots + A(\mu_1\mu_2 \ldots \mu_m) + \ldots$$

then $$(1^{\mu_1}))1^{\mu_2}) \ldots (1^{\mu_m}) = \ldots + A(\lambda_1\lambda_2 \ldots \lambda_l) + \ldots$$

a theorem known to algebraists as the Cayley-Betti Law of Symmetry in Symmetric Functions.

The easy intuitive nature of this proof of the theorem is very remarkable.

Art. 5.—In the above the magnitude of the numbers, appearing in the compartments of the lattice, has been restricted so as not to exceed unity. This restriction may be removed in the following manner. Consider the symmetric functions known as the homogeneous product sums of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ viz.:—

$$h_1 = (1),$$
$$h_2 = (2) + (1^2),$$
$$h_3 = (3) + (21) + (1^3),$$
$$\ldots\ldots\ldots\ldots$$

and note the result

$$D_s h_\lambda = h_{\lambda-s},$$

and also $$D_s h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l} = \Sigma h_{\lambda_1-\sigma_1} h_{\lambda_2-\sigma_2} h_{\lambda_3-\sigma_3} \ldots h_{\lambda_l-\sigma_l}$$

where $(\sigma_1\,\sigma_2\,\sigma_3 \ldots \sigma_l)$ is a partition of s and the sum is for all such partitions, and for a particular partition is for all ways of operating upon the suffixes with the parts of the partition. Thus

$$\begin{aligned} D_3 h_\lambda h_\mu h_\nu = {} & h_{\lambda-3} h_\mu h_\nu + h_\lambda h_{\mu-3} h_\nu + h_\lambda h_\mu h_{\nu-3} \\ & + h_{\lambda-2} h_{\mu-1} h_\nu + h_\lambda h_{\mu-2} h_{\nu-1} + h_{\lambda-1} h_\mu h_{\nu-2} \\ & + h_{\lambda-1} h_{\mu-2} h_\nu + h_\lambda h_{\mu-1} h_{\nu-2} + h_{\lambda-2} h_\mu h_{\nu-1} \\ & + h_{\lambda-1} h_{\mu-1} h_{\nu-1}. \end{aligned}$$

If from the result of $D_s\, h_{\lambda_1}, h_{\lambda_2} \ldots h_{\lambda_l}$ we select the term product $h_{\lambda_1-\sigma_1}$, $h_{\lambda_2-\sigma_2} \ldots h_{\lambda_l-\sigma_l}$, the corresponding lattice will have as first row

σ_1	σ_2	σ_3	σ_4	.....	σ_l

the sum of the numbers being s, and if in the selected product we now operate with D_t we can select a term product from the result, and the two minor operations may be indicated by the two-row lattice,

σ_1	σ_2	σ_3	σ_4	.....	σ_l
τ_1	τ_2	τ_3	τ_4	.....	τ_l

the sum of the numbers τ being t.

Hence if we take as operation

$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m},$$

and as function $$h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l},$$

we will obtain a number of lattices of m rows and l columns, which possess the property that the sums of the numbers in the successive rows are $\mu_1, \mu_2, \ldots \mu_m$, and in the successive columns $\lambda_1, \lambda_2, \ldots \lambda_l$, no restriction being placed upon the magnitude of the numbers. The number of such lattices is A, where

$$h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_l} = \ldots + A(\mu_1 \mu_2 \ldots \mu_m) + \ldots$$

and now transposition of lattices shows that

$$h_{\mu_1} h_{\mu_2} \ldots h_{\mu_m} = \ldots + A(\lambda_1 \lambda_2 \ldots \lambda_l) + \ldots$$

yielding a proof of a law of symmetry discovered by the present author many years ago. The process involves the actual formation of the things enumerated by the number A. The secret of its success in this instance lies in the result $D_\lambda h_s = h_{s-\lambda}$.

Ex. gr. We have

$$D_2^2 D_1^2\, h_3 h_2 h_1 = 18$$

i.e.

$$h_3 h_2 h_1 = \ldots + 18(2211) + \ldots$$

$$h_2^2 h_1^2 = \ldots + 18(321) + \ldots$$

and we must have 18 lattices; now eight of these, in which the compartment numbers do not exceed unity, have been depicted above; the remaining 10 are

2		
	2	
1		
		1

2		
	2	
		1
1		

2		
1	1	
	1	
		1

2		
1	1	
		1
	1	

2		
	1	1
1		
	1	

2		
	1	1
	1	
1		

2		
1		1
	1	
	1	

	2	
2		
1		
		1

	2	
2		
		1
1		

	2	
1		1
1		
1		

Art. 6.—The next problem I will consider is that in which the magnitude of the compartment numbers has a superior limit k.

Let k_s denote the homogeneous product sum of order s in which none of the quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$ is raised to a higher power than k. *Ex. gr.* If $k = 2$, k_3 will be $(21) + (1^3)$, and not $(3) + (21) + (1^3)$.

We have $D_\lambda k_s = k_{s-\lambda}$ where $\lambda \leqq k$,

and $D_\lambda k_s = 0$ if $\lambda > k$.

Take as operation

$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m}$$

and as function $k_{\lambda_1}, k_{\lambda_2}, \ldots k_{\lambda_l}$, and we will obtain a number of lattices of m rows and l columns which possess the property that the sums of the numbers in the successive rows are $\mu_1, \mu_2, \ldots \mu_m$, and in the successive columns $\lambda_1, \lambda_2, \ldots \lambda_l$, the magnitude of the compartment numbers being restricted not to exceed k.

The number of such lattices is

$$D_{\mu_1} D_{\mu_2} \ldots D_{\mu_m}\, k_{\lambda_1} k_{\lambda_2} \ldots k_{\lambda_l} = A,$$

where

$$k_{\lambda_1} k_{\lambda_2} \ldots k_{\lambda_l} = \ldots + A(\mu_1 \mu_2 \ldots \mu_m) + \ldots,$$

and by transposing the lattices

$$k_{\mu_1} k_{\mu_2} \ldots k_{\mu_m} = \ldots + A(\lambda_1 \lambda_2 \ldots \lambda_l) + \ldots;$$

establishing a law of symmetry in symmetrical algebra.

I observe that if $k = 9$, the lattices associated with and enumerated by

$$D^3_{15} k^3_{15}$$

include all the row and column-magic squares connected with the natural series of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. In general if $N = \frac{1}{2}n(n^2 + 1)$, the lattices enumerated by $D^n_N\ k^n_N$, where $k = n^2$, include all the magic squares of order n connected with the first n^2 numbers. If we could further impress the condition that no compartment number is to be twice repeated, we would be successful in enumerating the magic squares divorced from the diagonal property. This seems to be a matter of difficulty, which is increased if an attempt be made to introduce diagonal and other conditions to which certain classes of magic squares are subject.

It may be gathered from what has been said, that every case of symmetric function multiplication is connected with a theory of lattice combinations. For if we take as function

$$(\lambda_1\mu_1\nu_1 \ldots)(\lambda_2\mu_2\nu_2 \ldots) \ldots (\lambda_s\mu_s\nu_s \ldots),$$

and as operation

$$D_p D_q D_r \ldots,$$

where

$$p + q + r + \ldots = \Sigma\lambda + \Sigma\mu + \Sigma\nu + \ldots,$$

we have

$$D_p D_q D_r \ldots (\lambda_1\mu_1\nu_1 \ldots ((\lambda_2\mu_2\nu_2 \ldots)(\ldots) \ldots (\lambda_s\mu_s\nu_s \ldots) = A,$$

where

$$(\lambda_1\mu_1\nu_1 \ldots)(\lambda_2\mu_2\nu_2 \ldots) \ldots (\lambda_s\mu_s\nu_s \ldots) = \ldots + A(pqr \ldots) + \ldots;$$

that is to say, we multiply together a number of monomial symmetric functions so as to exhibit it as a sum of monomial functions; in this sum we find a particular monomial function affected with a numerical coefficient A which, as shown by the present theory, is the number which enumerates lattices of a certain class easily definable. Thus, in the present instance, if the partition $(pqr \ldots)$ involve t parts, the lattices have s columns and t rows; the operation D_p acts, through its various partitions, upon the product of monomials, and any mode of picking out a partition of p from the factors of the product, one part from each factor, constitutes a minor operation which yields the first row of a lattice; the operation D_q is similarly responsible for all the second rows of the lattices, and finally every resulting lattice possesses a property which may be defined as under:—

The numbers in the successive rows are partitions of the numbers p, q, r, ... respectively, and in the successive columns are the partitions $(\lambda_1\mu_1\nu_1 \ldots)$, $(\lambda_2\eta_2\nu_2 \ldots)$, ... $(\lambda_s\mu_s\nu_s \ldots)$ respectively. Such are the lattices enumerated by the number A. One is reminded somewhat of CAYLEY's well-known algorithm for symmetric function multiplication (invented by him for use in his researches in the theory of Invariants), but here the determination is representative as well as enumerative, and has moreover analytical expression.

Ex. gr. Take as function (987)(654)(321), and as operation $D_{13}D_{15}D_{17}$; then

$$D_{13}D_{15}D_{17}(987)(654(321) = A,$$

where $$(987)(654)(321) = \ldots + A(13.15.17) + \ldots .$$

One of the associated lattices is

7	5	1
8	4	3
9	6	2

where observe that the numbers in the successive rows constitute partitions of the numbers 13, 15, 17 respectively, whilst in the successive columns the numbers constitute the partitions (987), (654), (321) respectively. The number of lattices possessing this property, is A, and A is readily found to have the value 6. If we had to find an expression for the number of row and column-magic squares of order 3, it would be necessary to write down the sum of all products

$$(762)(951)(843)$$

formed from the first 9 ($= n^2$) numbers in such wise that the content of each partition factor is $15 = \frac{1}{2}n(n^2 + 1)$, attention being paid to the order of the partitions, and to take as operation D_{15}^3 or in general $D^n_{\frac{1}{2}n(n^2+1)}$. The resulting lattices will all be magic squares in which the diagonal property is not essential, and the result of the operation upon the function will give the enumerating number.

Art. 7. To resume; in the lattice compartments we find invariably the numbers $\lambda_1, \mu_1, \nu_1, \ldots \lambda_2, \mu_2, \nu_2, \ldots \lambda_s, \mu_s, \nu_s, \ldots$ such numbers being subject to certain conditions for each row and each column. The assemblages of numbers in the successive columns do not vary from lattice to lattice, but those in the successive rows do vary from lattice to lattice.

Let $$\lambda_1 + \mu_1 + \nu_1 + \ldots = \lambda;\ \lambda_2 + \mu_2 + \nu_2 + \ldots = \mu;$$
$$\lambda_3 + \mu_3 + \nu_3 + \ldots = \nu, \text{ \&c.} \ldots$$

Then we have the following facts :—

(i.) The whole assemblage of numbers, $\lambda_1, \mu_1, \nu_1, \ldots \lambda_2, \mu_2, \nu_2, \ldots \lambda_3\ \mu_3\ \nu_3, \ldots$ is unaltered from lattice to lattice.

(ii.) The numbers $\lambda, \mu, \nu, \ldots$ appertaining to the columns, and the numbers $p_1\ q_1\ r, \ldots$ appertaining to the rows, are unaltered from lattice to lattice.

These conditions do not define the lattices in question, because other lattices comply with them, viz., those in which, the whole assemblages of compartment numbers remaining unchanged, the column partitions, while satisfying the condition (ii.), are other than

$$(\lambda_1\mu_1\nu_1 \ldots), (\lambda_2\mu_2\nu_2 \ldots), (\lambda_3\mu_3\nu_3 \ldots), \ldots$$

successively.

Let

$$\lambda_1' + \mu_1' + \nu_1' + \ldots = \lambda,$$
$$\lambda_2' + \mu_2' + \nu_2' + \ldots = \mu,$$
$$\lambda_3' + \mu_3' + \nu_3' + \ldots = \nu,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

the assemblage of dashed numbers being in some order identical with the assemblages of undashed numbers. The new conditions include lattices enumerated by

$$D_{pqr} \ldots (\lambda_1'\mu_1'\nu_1' \ldots)(\lambda_2'\mu_2'\nu_2' \ldots)(\lambda_3'\mu_3'\nu_3' \ldots) \ldots$$

and the totality of lattices, implied by them, is enumerated by

$$D_{pqr} \ldots \Sigma(\lambda_1'\mu_1'\nu_1' \ldots)(\lambda_2'\mu_2'\nu_2' \ldots)(\lambda_3'\mu_3'\nu_3' \ldots) \ldots$$

the summation being for every separation of the assemblage of numbers

$$\lambda_1, \mu_1, \nu_1, \ldots \lambda_2, \mu_2, \nu_2, \ldots \lambda_3, \mu_3, \nu_3 \ldots$$

into partitions

$$(\lambda_1'\mu_1'\nu_1' \ldots), (\lambda_2'\mu_2'\nu_2' \ldots), (\lambda_3'\mu_3'\nu_3' \ldots), \ldots$$

such that

$$\lambda_1' + \mu_1' + \nu_1' + \ldots = \lambda,$$
$$\lambda_2' + \mu_2' + \nu_2' + \ldots = \mu,$$
$$\lambda_3' + \mu_3' + \nu_3' + \ldots = \nu;$$

or, as it is convenient to say, for every separation of the given assemblage of numbers which has the *specification* $\lambda, \mu, \nu \ldots$ With this nomenclature we may say that the successive row partitions have a specification $p, q, r \ldots$ and we may assert that the lattices under enumeration are associated with a definite assemblage of numbers and with two specifications, all three of which denote partitions of the same number, $\Sigma\lambda' + \Sigma\mu' + \Sigma\nu' + \ldots = \lambda + \mu + \nu + \ldots = p + q + r + \ldots$ We thus associate the lattice with three partitions of one number.

There is a law of symmetry connected with these lattices the true nature of which is not at once manifest; it is *not* obtained by simple transposition of the above lattices, and we are *not* permitted to simply exchange the partitions $(\lambda\mu\nu \ldots)$, $(pqr \ldots)$ preserving the assemblage of compartment numbers with the object of obtaining identity of enumeration. The difficulty presents itself whenever two or more partitions, $(\lambda_1'\mu_1'\nu_1' \ldots)$, $(\lambda_2'\mu_2'\nu_2' \ldots)$, &c. . . . are *different* but have the *same specification*. I will obtain the true theorem by the examination of a particular case. Let the assemblage of numbers be 2, 2, 1, 1, and consider the two results

$$(2)^2(1)^2 = \ldots + 6(2^3) + \ldots$$

connected with $D_2^3(2)^2(1)^2 = 6$,

and $$(2)^2(1^2) = \ldots + 2(2^2 1^2) + \ldots$$

connected with $D_2^2 D_1^2 (2)^2 (1^2) = 2$.

In the first case the row and column specifications are 2, 2, 2, and 2, 2, 1, 1, respectively; and in the second case 2, 2, 1, 1, and 2, 2, 2, respectively.

The first case yields the six lattices

2			
	2		
		1	1

	2		
2			
		1	1

2			
		1	1
	2		

	2		
		1	1
2			

		1	1
2			
	2		

		1	1
	2		
2			

the second case the two lattices

2		
	2	
		1
		1

	2	
2		
		1
		1

If we transpose the six lattices we obtain four lattices in addition to these two, viz. :—

2		
		2
	1	
	1	

		2
2		
	1	
	1	

;

	2	
		2
1		
1		

		2
	2	
1		
1		

The first pair of these would be derived from $D_2^2 D_1^2 (2)(1^2)(2)$, and the second pair from $D_2^2 D_1^2 (1^2)(2)^2$. Hence it is clear that to obtain identity of enumeration we must multiply $(2)^2(1^2)$ by a number equal to the number of ways of permuting the factors which have the same specification, viz., by $\frac{(2+1)!}{2!\,1!} = 3$.

The corresponding multiplier of $(2)^2(1^2)$ is $\frac{2!}{2!} \cdot \frac{2!}{2!} = 1$.

Let then an operand be

$$(L_1)^{l_1}(L_2)^{l_2}(L_3)^{l_3} \ldots (M_1)^{m_1}(M_2)^{m_2}(M_3)^{m_3} \ldots$$

$L_1, L_2, L_3, \ldots$ denoting different partitions of the same weight,
$M_1, M_2, M_3 \ldots$ „ „ „
&c., &c., &c.

we attach a coefficient

$$\frac{(l_1 + l_2 + l_3 + \ldots)!}{l_1!\,l_2!\,l_3!\ldots} \cdot \frac{(m_1 + m_2 + m_3 \ldots)!}{m_1!\,m_2!\,m_3!\ldots} \ldots$$

Let any operand

$$(\lambda_1'\mu_1'\nu_1' \ . \ . \ .)(\lambda_2'\mu_2'\nu_2' \ . \ . \ .)(\lambda_3'\mu_3'\nu_3' \ . \ . \ .) \ . \ . \ .$$

so multiplied be denoted by

$$\text{Co}(\lambda_1'\mu_1'\nu_1' \ . \ . \ .)(\lambda_2'\mu_2'\nu_2' \ . \ . \ .)(\lambda_3'\mu_3'\nu_3' \ . \ . \ .) \ . \ . \ .$$

then we have the following law of symmetry :—

From a given finite assemblage of numbers

$$\lambda_1, \mu_1, \nu_1, \ . \ . \ ., \quad \lambda_2, \mu_2, \nu_2, \ . \ . \ ., \quad \lambda_3, \mu_3, \nu_3, \ . \ . \ . \quad . \ . \ .,$$

construct all the products

$$(\lambda_1'\mu_1'\nu_1' \ . \ . \ .)(\lambda_2'\mu_2'\nu_2' \ . \ . \ .)(\lambda_3'\mu_3'\nu_3' \ . \ . \ .) \ . \ . \ .$$

which have a given specification $(\lambda\mu\nu \ . \ . \ .)$ and all the products

$$(p_1q_1r_1 \ . \ . \ .)(p_2q_2r_2 \ . \ . \ .)(p_3q_3r_3 \ . \ . \ .) \ . \ . \ .$$

which have a given specification $(pqr \ . \ . \ .)$.

If

$$\Sigma\text{Co}(\lambda_1'\mu_1'\nu_1' \ . \ . \ .)(\lambda_2'\mu_2'\nu_2' \ . \ . \ .)(\lambda_3'\mu_3'\nu_3' \ . \ . \ .) \ . \ . \ . = \ . \ . \ . + \text{A}(pqr \ . \ . \ .) + \ . \ . \ .$$

then

$$\Sigma\text{Co}(p_1q_1r_1 \ . \ . \ .)(p_2q_2r_2 \ . \ . \ .)(p_3q_3r_3 \ . \ . \ .) \ . \ . \ . = \ . \ . \ . + \text{A}(\lambda\mu\nu \ . \ . \ .) + \ . \ . \ .$$

the lattices being derived from

$$\text{D}_{pqr\ldots}\,\Sigma\text{Co}(\lambda_1'\mu_1'\nu_1' \ . \ . \ .)(\lambda_2'\mu_2'\nu_2' \ . \ . \ .)(\lambda_3'\mu_3'\nu_3' \ . \ . \ .) \ . \ . \ . = \text{A}$$
$$\text{D}_{\lambda\mu\nu\ldots}\,\Sigma\text{Co}(p_1q_1r_1 \ . \ . \ .)(p_2q_2r_2 \ . \ . \ .)(p_3q_3r_3 \ . \ . \ .) \ . \ . \ . = \text{A}.$$

This is the most refined law of symmetry that has yet come to light in the algebra of a single system of quantities (*cf.* "Memoirs on Symmetric Functions," 'Amer. J.,' *loc. cit.*). The actual representation of the things enumerated by the number A is obtained with ease by this theory of the lattice.

§ 2.

Art. 8.—So far the operations have been those of the infinitesimal calculus, and the numbers involved in the partitions of the functions have been positive integers excluding zero. If we admit zero as a part in the partitions, we find that we have to do with the operations of the calculus of finite differences. At the commencement of the paper d/dx was shown to be a combinatorial symbol, in that when operating upon a power of x, the said power being positive and integral, it had the effect of summing the results obtained by substituting unity for x in all possible ways in the product of x's. Now the corresponding operator of the calculus of finite differences, viz., Δ operates upon a power of x by striking out one x, two x's, three x's, &c., in all possible ways and summing the results. Thus

$$\Delta x^3 = \not{x}xx + x\not{x}x + xx\not{x} + x\not{x}\not{x} + \not{x}x\not{x} + \not{x}\not{x}x + \not{x}\not{x}\not{x} = 3x^2 + 3x + 1.$$

This simple fact shows that we may expect a corresponding theory of lattices, and that this is, in fact, the case is seen immediately one introduces the part zero into the partitions of the functions. I have introduced zero parts into partitions in the Memoirs on Symmetric Functions above alluded to, and have imported into the theory the corresponding operators d_0 and D_0.* It was there shown that, if n be the number of quantities of which the symmetric functions are formed,

$$d_0 = \frac{d}{dn}, \quad D_0 = e^{\frac{d}{dn}} - 1\ ;$$

and thence it appears that we have operations D_0, d_0, $1 + D_0$ corresponding to the operations Δ, d/dn, E of the calculus of finite differences.

Considering partitions which only involve zero parts, we have only finite difference operations; if we have other integers, we have mixed operations drawn both from the finite and infinitesimal calculus.

The partition (0^p) is derived from

$$\Sigma\alpha_1{}^q\alpha_2{}^q\alpha_3{}^q \ . \ . \ . \ \alpha_p{}^q$$

by putting $q = 0$, and obviously has the value $\binom{n}{p}$ and, in the paper referred to, it has been shown that D_0 operates upon a monomial by erasing one zero part from its partition, so that

$$D_0(0^p) = (0^{p-1}),$$

which is to be compared with the operation of Δ, viz.:—

$$\Delta x^{(m)} = mx^{(m-1)}$$

where $$x^{(m)} = x(x-1)(x-2) \ . \ . \ . \ (x-m+1)$$

in the notation of the finite calculus.

Further, it has been shown that D_0 operates upon a product of monomials through its partitions 0, 00, 000, 0000, . . . which are infinite in number, viz.:—we are to strike out one zero, two zeros, three zeros, &c., in all possible ways; but in any one such operation not striking out more than one zero from any monomial factor.

Ex. gr.
$$\begin{aligned} D_0(0^3)(0^2)(0) = {} & (0^2)(0^2)(0) + (0^3)(0)(0) + (0^3)(0^2) \\ & + (0^2)(0)(0) + (0^2)(0^2) + (0^3)(0) \\ & + (0^2)(0) \end{aligned}$$

the successive lines being due to the partitions 0, 00, 000 respectively.

* 'American Journal of Mathematics,' vol. 12, second memoir, "On a New Theory of Symmetric Functions," p. 71 *et seq.*; and vol. 13, third memoir, &c., pp. 8 *et seq.*

Compare the difference formula

$$\Delta u_x v_x w_x = (\mathrm{EE'E''} - 1)\, u_x v_x w_x$$
$$= (\Delta + \Delta' + \Delta'' + \Delta'\Delta'' + \Delta''\Delta + \Delta\Delta' + \Delta\Delta'\Delta'')\, u_x v_x w_x,$$

where u_x is only operated upon by E and Δ, v_x by E′ and Δ', w_x by E″ and Δ''.

Art. 9.—Consider the lattice theory connected with the operation D_0 and zero-part partition functions.

Take the function

$$(0^\lambda)(0^\mu)(0^\nu)\ \ldots$$

$\lambda, \mu, \nu, \ldots$ being in descending order; if it be multiplied out, it will appear as a linear function of (0^λ), $(0^{\lambda+1})$, . . . $(0^{\lambda+\mu+\nu+\ldots})$, the coefficients being positive (*cf.* Second Memoir, *loc. cit.*, p. 102).

To find therein the coefficient of the term (0^s) we must operate with $D_0^{\ s}$, and the sought coefficient is the resulting *numerical* term. If the factors $(0^\lambda)(0^\mu)(0^\nu)\ldots$ be t in number, we are concerned with lattices of t columns and s rows. The first operation of D_0 results in a first row whose compartments contain t or fewer zeros placed in any manner so that not more than one zero is in each compartment; similarly, for the successive rows and the final lattice is subject to the single condition that the numbers of zeros in the successive columns are $\lambda, \mu, \nu, \ldots$ respectively. The number of such lattices is

$$\{D_0^s (0^\lambda)(0^\mu)(0^\nu)\ \ldots\}_{n=0} = \{(e^{\frac{d}{dn}} - 1)^s \tbinom{n}{\lambda}\tbinom{n}{\mu}\tbinom{n}{\nu}\ \ldots\}_{n=0}$$

or, symbolically,
$$(e^{\frac{d}{do}} - 1)^s \tbinom{0}{\lambda}\tbinom{0}{\mu}\tbinom{0}{\nu}\ \ldots$$

We have thus the analytical solution of a distribution problem upon a lattice.

It may be convenient to give the lattice a literal form by writing α for zero in the compartments.

Art. 10.—Contrast the result obtained with that which arises from

$$D_1^s (1^\lambda)(1^\mu)(1^\nu)\ \ldots$$

The lattices are similar to those above, with the additional condition that each row is to contain but one letter α. Again, from

$$D_{p_1} D_{p_2} \ldots D_{p_s} (1^\lambda)(1^\mu)(1^\nu)\ \ldots$$

arise lattices of t columns and s rows with the same condition as the zero lattices, but with the additional conditions that the numbers of letters α in the successive rows are to be $p_1, p_2, \ldots p_s$ respectively. This remark leads to a relationship between the coefficients in the developments of $(1^\lambda)(1^\mu)(1^\nu)\ldots$ and $(0^\lambda)(0^\mu)(0^\nu)\ldots$ respectively. For let

$$(1^\lambda)(1^\mu)(1^\nu)\ \ldots = \ldots + A_{p_1 p_2 \ldots p_s}(p_1 p_2 \ldots p_s)$$
$$+ A_{p_1' p_2' \ldots p_s'}(p_1' p_2' \ldots p_s') + \ldots,$$

the terms written comprising all monomial functions whose partitions contain exactly s parts; and

$$(0^\lambda)(0^\mu)(0^\nu)\ldots = \ldots + B_s(0^s) + \ldots$$

If $P_{p_1p_2\ldots p_s}$ denote the number of permutations of the numbers $p_1, p_2, \ldots p_s$, I say that the above lattice theory establishes the relation

$$B_s = P_{p_1p_2\ldots p_s} A_{p_1p_2\ldots p_s} + P_{p_1'p_2'\ldots p_s'} A_{p_1'p_2'\ldots p_s'} + \ldots$$

Ex. gr. Observe the two results

$$(1^2)(1)(1) = (31) + 2(2^2) + 5(21^2) + 12(1^4),$$
$$(0^2)(0)(0) = 4(0^2) + 15(0^3) + 12(0^4),$$

and verify that the relation given obtains between the coefficients.

Art. 11.—This zero theory is really nothing more than a calculus of binomial coefficients, which enables the study of their properties by means of the powerful instruments appertaining to the Theory of Symmetric Functions. The Law of Symmetry established by the author in the Second Memoir (*loc. cit.*) is easily established by means of the lattice; it may be stated in a simple form, because all the functions $(0), (0^2), (0^3), \ldots$ are to be regarded as having the same specification, viz. (0); further, the specification of $(0^\lambda)(0^\mu)(0^\nu)\ldots$ to s factors is (0^s)). Select all the products formed from a *given number* of zeros which have the same specification (0^i), and attach to each a coefficient equal to the number of permutations of which it is susceptible. Denote the sum of such products by $\Sigma \text{Co}\,(0^\lambda)(0^\mu)(0^\nu)\ldots$. Similarly, for a specification (0^j) denote the sum of such products by $\Sigma \text{Co}\,(0^p)(0^q)(0^r)\ldots$. Then

$$\Sigma \text{Co}(0^\lambda)(0^\mu)(0^\nu)\ldots = \ldots + A(0^j) + \ldots$$
$$\Sigma \text{Co}(0^p)(0^q)(0^r)\ldots = \ldots + A(0^i) + \ldots$$

the coefficient A being the same in both cases.

Ex. gr. Verify that

$$2(0^3)(0) + (0^2)^2 = \ldots + 12(0^3) + \ldots$$
$$3(0^2)(0)^2 = \ldots + 12(0^2) + \ldots$$

§ 3.

Art. 12.—I continue the general plan of this paper, viz.:—I do not attempt the solution of any particular problems, unless they are suggested by the general course of the investigation, but rather start with definite operations and functions, and seek to discover the problems of which they furnish the solution. This is the reverse process to that employed in the 'Trans. Camb. Phil. Soc.' (*loc. cit.*), where I particularly investigated a number of questions more or less directly associated with the famous Problem of the Latin Square. I anticipate what follows to the extent of observing that the Latin Square again presents itself without special effort on the part of the investigator, and that a new and very simple solution of that and associated problems is obtained.

I seek to obtain theorems which flow from a consideration of symmetric functions of several systems of quantities, taken in conjunction with appropriate operations.

I make the reference MACMAHON, "Memoir on the Roots of Systems of Equations," 'Phil. Trans.,' A, 1890.

Consider the systems of quantities

$$\begin{array}{l} \alpha_1, \alpha_2, \alpha_3, \ldots \\ \beta_1, \beta_2, \beta_3, \ldots \\ \gamma_1, \gamma_2, \gamma_3, \ldots \\ \ldots\ldots\ldots \end{array}$$

and write

$$(1 + \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 + \ldots)(1 + \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3) \ldots$$
$$= 1 + a_{100\ldots} x_1 + a_{010\ldots} x_2 + \ldots + a_{pqr\ldots} x_1^p x_2^q x_3^r \ldots + \ldots$$

Denote the symmetric function

$$\Sigma \alpha_1^{p_1} \beta_1^{q_1} \gamma_1^{r_1} \ldots \alpha_2^{p_2} \beta_2^{q_2} \gamma_2^{r_2} \ldots \alpha_3^{p_3} \beta_3^{q_3} \gamma_3^{r_3} \ldots \quad \ldots$$

by $\qquad (\overline{p_1 q_1 r_1 \ldots}\ \overline{p_2 q_2 r_2 \ldots}\ \overline{p_3 q_3 r_3 \ldots} \quad \ldots)$

so that $\qquad a_{pqr\ldots} = (\overline{100\ldots}^p\ \overline{010\ldots}^q\ \overline{001\ldots}^r \quad \ldots)$

Ex. gr. $\qquad a_{1111} = \Sigma \alpha_1 \beta_2 \gamma_3 \delta_4 = (\overline{1000}\ \overline{0100}\ \overline{0010}\ \overline{0001})$

The quantities $a_{pqr\ldots}$ are the elementary symmetric functions.

The linear operator $d_{\pi\kappa\rho\ldots}$ is defined by

$$d_{\pi\kappa\rho\ldots} = \overset{p,q,r,\ldots}{\Sigma} a_{p-\pi,\, q-\kappa,\, r-\rho,\ldots} \frac{d}{da_{pqr\ldots}},$$

so that

$$d_{100\ldots} = \Sigma\, a_{p-1,\, q,\, r\ldots} \frac{d}{da_{pqr\ldots}},$$

$$d_{010\ldots} = \Sigma\, a_{p,\, q-1,\, r\ldots} \frac{d}{da_{pqr\ldots}},$$

$$d_{001\ldots} = \Sigma\, a_{p,\, q,\, r-1,\ldots} \frac{d}{da_{pqr\ldots}},$$

and then $\qquad D_{pqr\ldots} = \dfrac{1}{p!\,q!\,r!\ldots} d_{100\ldots}^p\, d_{010\ldots}^q\, d_{001\ldots}^r \ldots,$

the multiplication of operators being symbolic as in TAYLOR'S theorem, so that $D_{pqr\ldots}$ is an operator of the order $p + q + r + \ldots$ and does *not* denote

$p+q+r+\ldots$ successive linear operations. The operation of $D_{pqr\ldots}$ upon a monomial symmetric function has been explained (*loc. cit.*). It has the effect of obliterating a part $\overline{pqr\ldots}$ from the partition of the function when such a part is present, and an annihilating effect is every other case. The operation upon a product has the effect of erasing a partition of $\overline{pqr\ldots}$ from the product, one part from each factor in all possible ways, the result of the operation being a sum of products, one product arising from each erasure of a partition.

Ex. gr. $$D_{43}(\overline{43}\ \overline{22}) = (\overline{22}).$$

If we have to operate with D_{43} upon

$$(\overline{32}\ \overline{22})\,(\overline{21}\ \overline{11})$$

we have to erase the two partitions $(\overline{32}\ \overline{11})$, $(\overline{22}\ \overline{21})$, and arrive at

$$D_{43}(\overline{32}\ \overline{22})\,(\overline{21}\ \overline{11}) = (\overline{22})\,(\overline{21}) + (\overline{32}\ \overline{11}).$$

Art. 13.—It will suffice to consider three systems of quantities as typical of the general case.

Take the function

$$a_{\lambda_1\mu_1\nu_1}\,a_{\lambda_2\mu_2\nu_2}\ldots a_{\lambda_s\mu_s\nu_s} = (\overline{100}^{\lambda_1}\,\overline{010}^{\mu_1}\,\overline{001}^{\nu_1})\,(\overline{100}^{\lambda_2}\,\overline{010}^{\mu_2}\,\overline{001}^{\nu_2})\ldots(\overline{100}^{\lambda_s}\,\overline{010}^{\mu_s}\,\overline{001}^{\nu_s})$$

and the operation $$D_{p_1q_1r_1}\,D_{p_2q_2r_2}\ldots D_{p_tq_tr_t}$$

$$(\overline{\lambda_1\mu_1\nu_1}\ \overline{\lambda_2\mu_2\nu_2}\ldots\overline{\lambda_s\mu_s\nu_s}) \text{ and } (\overline{p_1q_1r_1}\ \overline{p_2q_2r_2}\ldots\overline{p_tq_tr_t})$$

being each partitions of the same tripartite number.

If $$a_{\lambda_1\mu_1\nu_1}\,a_{\lambda_2\mu_2\nu_2}\ldots a_{\lambda_s\mu_s\nu_s} = \ldots + A(p_1q_1r_1\ p_2q_2r_2\ldots p_tq_tr_t) + \ldots,$$
$$D_{p_1q_1r_1}D_{p_2q_2r_2}\ldots D_{p_tq_tr_t}\ a_{\lambda_1\mu_1\nu_1}\,a_{\lambda_2\mu_2\nu_2}\ldots a_{\lambda_s\mu_s\nu_s} = A;$$

and we have to determine the nature of the lattices enumerated by the number A. The tripartite number $(\overline{p_1q_1r_1})$ has a partition of $p_1+q_1+r_1$ parts, viz.:— $\overline{100}^{p_1}$, $\overline{010}^{q_1}$, $\overline{001}^{r_1}$ so that, in operating with $D_{p_1q_1r_1}$ upon the operand, we have to select this partition from the product, one part from each factor, in all possible ways; the operation breaks up into minor operations as usual, and the first row of the lattice of s columns and t rows will contain in $p_1+q_1+r_1$ of its compartments the tripartite numbers 100, 010, 001 (p_1 of the first, q_1 of the second, and r_1 of the third) in some order; the assemblage of numbers in this row is the partition of the elementary function $a_{p_1q_1r_1}$. Similarly a minor operation of $D_{p_2q_2r_2}$ produces a second row containing tripartite numbers, the assemblage of which constitutes the elementary function $a_{p_2q_2r_2}$. We finally arrive at a lattice such that the tripartites in the successive rows constitute the elementary functions $a_{p_1q_1r_1}$, $a_{p_2q_2r_2}$, ... $a_{p_tq_tr_t}$ respectively, and in the successive columns the elementary functions $a_{\lambda_1\mu_1\nu_1}$, $a_{\lambda_2\mu_2\nu_2}$, . . . $a_{\lambda_s\mu_s\nu_s}$ respectively, and the number A enumerates the lattices possessing this property.

We may give this case a purely literal form by writing $100 = a$, $010 = b$, $001 = c$, and then we have a lattice of s columns and t rows, such that the products of letters in the successive rows are $a^{p_1}b^{q_1}c^{r_1}$, $a^{p_2}b^{q_2}c^{r_2}$, . . . $a^{p_t}b^{q_t}c^{r_t}$ respectively, and in the successive columns $a^{\lambda_1}b^{\mu_1}c^{\nu_1}$, $a^{\lambda_2}b^{\mu_2}c^{\nu_2}$, . . . $a^{\lambda_s}b^{\mu_s}c^{\nu_s}$ respectively.

Art. 14.—Stated in this form the problem appears to have a close relationship to the problem of the Latin Square. It is in fact a new generalization of that problem; for put $s = t = 3$ and

$$p_1 = q_1 = r_1 = p_2 = q_2 = r_2 = p_3 = q_3 = r_3 = 1$$
$$\lambda_1 = \mu_1 = \nu_1 = \lambda_2 = \mu_2 = \nu_2 = \lambda_3 = \mu_3 = \nu_3 = 1$$

so that the operation is D^3_{111} and the function a^3_{111}. One lattice is then

010	100	001
001	010	100
100	001	010

or in literal form

b	a	c
c	b	a
a	c	b

which is a Latin Square. Hence the numbers of Latin Squares of order 3 is

$$D^3_{111}\ a^3_{111},$$

and in general of order n

$$D^n_{111\ldots1}\ a^n_{111\ldots1},$$

a very simple solution of the problem. If reference be made to the solution arrived at (*loc. cit.*) by considerations relating to a single system of quantities, it will be noticed that the peculiar difficulties intrinsically present in that solution disappear at once when n systems of quantities are brought in as auxiliaries. The Latin Square appears at the outset of this investigation, and in a perfectly natural manner.

Art. 15.—Now put

$$p_1 = p_2 = \ . \ . \ . = p_{\lambda+\mu+\nu} = \lambda = \lambda_1 = \lambda_2 = \ . \ . \ . = \lambda_{\lambda+\mu+\nu}$$
$$q_1 = q_2 = \ . \ . \ . = q_{\lambda+\mu+\nu} = \mu = \mu_1 = \mu_2 = \ . \ . \ . = \mu_{\lambda+\mu+\nu}$$
$$r_1 = r_2 = \ . \ . \ . = r_{\lambda+\mu+\nu} = \nu = \nu_1 = \nu_2 = \ . \ . \ . = \nu_{\lambda+\mu+\nu},$$

so that $s = t = \lambda + \mu + \nu$. We have then lattices enumerated by

$$D^{\lambda+\mu+\nu}_{\lambda\mu\nu}\ a^{\lambda+\mu+\nu}_{\lambda\mu\nu},$$

and, in the literal form, they are such that the product of letters in each of the $\lambda + \mu + \nu$ rows and $\lambda + \mu + \nu$ columns is $a^\lambda b^\mu c^\nu$, one letter appearing in each compartment of the lattice. This is the extension of the idea of the Latin Square which was successfully considered in the former paper (*loc. cit.*), but now the enumeration is given in a simpler form and from simpler considerations.

In general, it has been established above that the Latin Squares based upon the product

$$a^\lambda b^\mu c^\nu \; . \; . \; .$$

are enumerated by the expression

$$D_{\lambda\mu\nu\,.\,.\,.}^{\lambda+\mu+\nu+\,.\,.\,.} \qquad a_{\lambda\mu\nu\,.\,.\,.}^{\lambda+\mu+\nu+\,.\,.\,.}$$

the simplicity of which leaves nothing to be desired.

Art. 16.—Consider the particular case of the general theorem which is such that no compartment is empty; the lattice has n columns and m rows.

$$p_1 + q_1 + r_1 = p_2 + q_2 + r_2 = \; . \; . \; . = p_m + q_m + r_m = n$$
$$\lambda_1 + \mu_1 + \nu_1 = \lambda_2 + \mu_2 + \nu_2 = \; . \; . \; . = \lambda_n + \mu_n + \nu_n = m,$$

the corresponding lattices being enumerated by

$$D_{p_1q_1r_1} D_{p_2q_2r_2} \; . \; . \; . \; D_{p_mq_mr_m} \quad a_{\lambda_1\mu_1\nu_1} a_{\lambda_2\mu_2\nu_2} \; . \; . \; . \; a_{\lambda_n\mu_n\nu_n},$$

These, when given the literal form, possess the property that the products of letters in the successive rows are $a^{p_1}b^{q_1}c^{r_1}$, $a^{p_2}b^{q_2}c^{r_2}$, . . . $a^{p_m}b^{q_m}c^{r_m}$ respectively, and in the successive columns $a^{\lambda_1}b^{\mu_1}c^{\nu_1}$, $a^{\lambda_2}b^{\mu_2}c^{\nu_2}$, . . . $a^{\lambda_n}b^{\mu_n}c^{\nu_n}$ respectively.

Ex. gr. Suppose the row products to be a^3b, a^2b^2, a^2b^2, a^4, and the column products a^4, a^3b, a^3b, ab^3

$$\begin{aligned} & D_{31}D_{22}^2D_{40}a_{40}a^2_{31}a_{13} \\ &= D_{31}D_{22}^2D_{40}(\overline{10}^4)(\overline{10}^3\,\overline{01})^2(\overline{10}\,\overline{01}^3) \\ &= D_{31}D_{22}^2(\overline{10}^3)(\overline{10}^2\,\overline{01})^2(\overline{01}^3) \\ &= D_{22}^2(\overline{10}^2)(\overline{10}\,\overline{01})^2(\overline{01}^2) \\ &= D_{22}\,2(\overline{10})^2(\overline{01})^2 \\ &= 2. \end{aligned}$$

and the two lattices are

a	*a*	*a*	*b*
a	*a*	*b*	*b*
a	*b*	*a*	*b*
a	*a*	*a*	*a*

a	*a*	*a*	*b*
a	*b*	*a*	*b*
a	*a*	*b*	*b*
a	*a*	*a*	*a*

Again, suppose the row products to be a^3b, a^2b^2, a^2b^2, and the column products a^3, a^2b, a^2b, b^3

$$
\begin{aligned}
&D_{31}D_{22}^2\, a_{30}\, a_{21}^2\, a_{03} \\
&= D_{31}D_{22}^2\, (\overline{10}^3)(\overline{10}^2\,\overline{01})^2(\overline{01}^2) \\
&= D_{22}^2\, (\overline{10}^2)(\overline{10}\,\overline{01})^2(\overline{01}^2) \\
&= 2D_{22}\, (\overline{10})^2(\overline{01})^2 = 2,
\end{aligned}
$$

and the lattices are

a	a	a	b
a	a	b	b
a	b	a	b

a	a	a	b
a	b	a	b
a	a	b	b

Art. 17.—In general, we may state that the lettered lattices, which are such that the row products are in order

$$a^{p_1}b^{q_1}c^{r_1}\ldots k^{z_1},\ a^{p_2}b^{q_2}c^{r_2}\ldots k^{z_2},\ \ldots\ a^{p_m}b^{q_m}c^{r_m}\ldots k^{z_m},$$

and the column products in order

$$a^{\lambda_1}b^{\mu_1}c^{\nu_1}\ldots k^{\zeta_1},\ a^{\lambda_2}b^{\mu_2}c^{\nu_2}\ldots k^{\zeta_2},\ \ldots\ a^{\lambda_n}b^{\mu_n}c^{\nu_n}\ldots k^{\zeta_n},$$

one letter being in each compartment, are enumerated by

$$D_{p_1q_1r_1\ldots z_1}D_{p_2q_2r_2\ldots z_2}\ldots D_{p_mq_mr_m\ldots z_m}\,a_{\lambda_1\mu_1\nu_1\ldots\zeta_1}\,a_{\lambda_2\mu_2\nu_2\ldots\zeta_2}\ldots a_{\lambda_n\mu_n\nu_n\ldots\zeta_n};$$

a very interesting development of the Latin Square problem.

We have found above the nature of the lattices enumerated by the number

$$D_{p_1q_1r_1\ldots}D_{p_2q_2r_2\ldots}\ldots D_{p_tq_tr_t}\ldots a_{\lambda_1\mu_1\nu_1\ldots}a_{\lambda_2\mu_2\nu_2\ldots}\ldots a_{\lambda_s\mu_s\nu_s\ldots}$$

any number of systems of quantities being involved, and the mere fact of the existence of the lattices indicates a law of symmetry which may be stated as follows :—

If

$$a_{\lambda_1\mu_1\nu_1\ldots}a_{\lambda_2\mu_2\nu_2\ldots}\ldots a_{\lambda_s\mu_s\nu_s\ldots} = \ldots + A(\overline{p_1q_1r_1\ldots}\,\overline{p_2q_2r_2\ldots}\ \ldots\overline{p_tq_tr_t\ldots}) + \ldots$$

then

$$a_{p_1q_1r_1\ldots}a_{p_2q_2r_2\ldots}\ldots a_{p_tq_tr_t\ldots} = \ldots + A(\overline{\lambda_1\mu_1\nu_1\ldots}\,\overline{\lambda_2\mu_2\nu_2\ldots}\ \ldots\overline{\lambda_s\mu_s\nu_s\ldots}) + \ldots$$

Art. 18.—The next case that comes forward for examination is that connected with the homogeneous product sums $h_{\lambda\mu\nu}\ldots$ We require the theorem

$$D_{pqr\ldots}h_{\lambda\mu\nu\ldots} = h_{\lambda-p,\,\mu-q,\,\nu-r\ldots}$$

and also

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$$D_{pqr\ldots} h_{\lambda_1\mu_1\nu_1\ldots} h_{\lambda_2\mu_2\nu_2\ldots} \ldots \ldots h_{\lambda_s\mu_s\nu_s\ldots},$$
$$= \Sigma h_{\lambda_1-p_1,\,\mu_1-q_1,\,\nu_1-r_1,\ldots} h_{\lambda_2-p_2,\,\mu_2-q_2,\,\nu_2-r_2,\ldots} \ldots \ldots h_{\lambda_s-p_s,\,\mu_s-q_s,\,\nu_s-r_s,\ldots}$$

where $(\overline{p_1q_1r_1\ldots}\ \overline{p_2q_2r_2\ldots}\ \ldots\ \overline{p_sq_sr_s\ldots})$ is a partition of $(\overline{pqr\ldots})$, and the sum is for all such partitions and for a particular partition is for all ways of operating upon the suffixes with the parts of the partition. *Ex. gr.*

$$D_{11}h_{11}h_{22} = h_{22} + h^2_{11} + h_{01}h_{21} + h_{10}h_{12}$$

Taking only tripartite functions for convenience, consider the function

$$h_{\lambda_1\mu_1\nu_1} h_{\lambda_2\mu_2\nu_2} \ldots h_{\lambda_s\mu_s\nu_s}$$

and the operation $D_{p_1q_1r_1} D_{p_2q_2r_2} \ldots D_{p_tq_tr_t}$;

we have $D_{p_1q_1r_1} D_{p_2q_2r_2} \ldots D_{p_tq_tr_t} h_{\lambda_1\mu_1\nu_1} h_{\lambda_2\mu_2\nu_2} \ldots h_{\lambda_s\mu_s\nu_s} = A$;

where $(\overline{p_1q_1r_1}\ \overline{p_2q_2r_2}\ \ldots\ \overline{p_tq_tr_t})$ and $(\overline{\lambda_1\mu_1\nu_1}\ \overline{\lambda_2\mu_2\nu_2}\ \ldots\ \overline{\lambda_s\mu_s\nu_s})$

being partitions of the same tripartite number,

$$h_{\lambda_1\mu_1\nu_1} h_{\lambda_2\mu_2\nu_2} \ldots h_{\lambda_s\mu_s\nu_s} = \ldots + A(\overline{p_1q_1r_1}\ \overline{p_2q_2r_2} \ldots \overline{p_tq_tr_t}) + \ldots$$

The operation of $D_{p_1q_1r_1}$ upon the product splits up as usual into a number of minor operations, one of which, as shown above, is connected with one of its partitions operating in a definite manner upon the suffixes $\lambda_1\mu_1\nu_1$, $\lambda_2\mu_2\nu_2$, ... $\lambda_s\mu_s\nu_s$. Hence the first row of the lattice has in certain of its s compartments the tripartite parts of some partition of $\overline{p_1q_1r_1}$; the second row also will have in certain of its compartments the tripartite parts of some partition of $\overline{p_2q_2r_2}$; and finally we must arrive at a lattice whose rows are associated with partitions of $\overline{p_1q_1r_1}$, $\overline{p_2q_2r_2}$, ... $\overline{p_tq_tr_t}$ respectively, and whose t columns are associated with partitions of $\overline{\lambda_1\mu_1\nu_1}$, $\overline{\lambda_2\mu_2\nu_2}$, ... $\overline{\lambda_s\mu_s\nu_s}$ respectively. There is no restriction on the magnitude of the constituents of the various tripartite numbers which appear in the compartments. The lattices thus defined are enumerated by the number A. We may give the lattice a literal form by writing $a^\rho b^\sigma c^\tau$ for $(\rho\sigma\tau)$ in a compartment. We then have a theorem which may be stated as follows :—

Monomial products of letters a, b, c, ... may be placed in the compartments of a lattice of t rows and s columns in such wise that the multiplication of products in successive rows produces $a^{p_1}b^{q_1}c^{r_1}\ldots$, $a^{p_2}b^{q_2}c^{r_2}\ldots$, ... $a^{p_t}b^{q_t}c^{r_t}\ldots$ respectively, and in successive columns produces $a^{\lambda_1}b^{\mu_1}c^{\nu_1}\ldots$, $a^{\lambda_2}b^{\mu_2}c^{\nu_2}\ldots$, ... $a^{\lambda_s}b^{\mu_s}c^{\nu_s}\ldots$ respectively in a number of ways enumerated by the number A above defined.

It is scarcely necessary to observe that

$$\Sigma p - \Sigma\lambda = \Sigma q - \Sigma\mu = \Sigma r - \Sigma\nu = \ldots = 0,$$

and that only $s + t - 1$ of the $s + t$ literal products are independent.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Art. 19.—In conclusion, it may be remarked that there is no difficulty in evolving a mixed theory which involves the operation both of the infinitesimal calculus and of the finite calculus. Operations and functions may be designed which lead to lattices which are not rectangular. The theory may be connected with complete or incomplete lattices in three or more dimensions; and finally one of the most promising paths of research appears to be connected with a multipartite zero. These matters may be the subjects of future investigation. For the present enough has been said to indicate the apparent scope of the new method.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES